

TRACE OPERATORS ON WIENER AMALGAM SPACES

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ABSTRACT. The paper deals with trace operators of Wiener amalgam spaces using frequency-uniform decomposition operators and maximal inequalities, obtaining sharp results. Additionally, we provide the embeddings between standard and anisotropic Wiener amalgam spaces.

1. INTRODUCTION

The aim of this paper is to study the trace problem: What can be said about the trace operator \mathbb{T} ,

$$\mathbb{T} : f(x) \rightarrow f(\bar{x}, 0), \quad \bar{x} = (x_1, x_2, \dots, x_{n-1}),$$

as a mapping from $W_s^{p,q}(\mathbb{R}^n)$ to $W_s^{p,q}(\mathbb{R}^{n-1})$. We note that for a tempered distribution f defined on \mathbb{R}^n , $f(x, 0)$ has no straightforward meaning and the question is how to define the trace for a class of tempered distributions. One can resort to the Schwartz function ϕ , which has a pointwise trace $\phi(\bar{x}, 0)$. It can be extended to (quasi-)Banach function spaces which contain the Schwartz space \mathcal{S} as a dense subspace.

Our setting is on Wiener amalgam spaces. These spaces, together with modulation spaces, were introduced by Feichtinger [5, 6, 7] in the 80's and are now widely used function spaces for various problems in PDE and harmonic analysis [1, 2, 3, 11, 16]. They resemble Triebel-Lizorkin spaces in the sense that we are taking $L^p(\ell^q)$ norms, but differ with the decomposition operator being used. Instead of the dyadic decomposition operators $\Delta_k \sim \mathcal{F}^{-1} \chi_{\{|\xi| \sim 2^k\}} \mathcal{F}$ used for Triebel-Lizorkin spaces, Wiener amalgam spaces use frequency uniform decomposition operators $\square_k \sim \mathcal{F}^{-1} \chi_{Q_k} \mathcal{F}$, where Q_k denotes a unit cube with center k and $\cup_{k \in \mathbb{Z}^n} Q_k = \mathbb{R}^n$.

The concept of trace operator plays an important role in studying the existence and uniqueness of solutions to boundary value problems, that is, to partial differential equations with prescribed boundary conditions [4, 14]. The trace operator makes it possible to extend the notion of restriction of a function to the boundary of its domain to "generalized" functions in various function spaces with regularity. Now, we give a formal definition for the trace operators.

Definition 1.1. Let X and Y be quasi-Banach function spaces defined on \mathbb{R}^n and \mathbb{R}^{n-1} , respectively. Assume that the Schwartz class \mathcal{S} is dense in X . Denote

$$\mathbb{T} : f(x) \rightarrow f(\bar{x}, 0), \quad f \in \mathcal{S}.$$

Assuming that there exist a constant $C > 0$ such that

$$\|\mathbb{T}f\|_Y \leq C\|f\|_X, \quad \forall f \in \mathcal{S},$$

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one can extend $\mathbb{T} : X \rightarrow Y$ by the density of \mathcal{S} in X and we write $f(\bar{x}, 0) = \mathbb{T}f$, which is said to be the trace of $f \in X$. Moreover, if there exist a continuous linear operator $\mathbb{T}^{-1} : Y \rightarrow X$ such that $\mathbb{T}\mathbb{T}^{-1}$ is the identity operator on Y , then \mathbb{T} is said to be a trace-retraction from X onto Y .

For $(\alpha-)$ modulation spaces, Besov spaces and Triebel-Lizorkin spaces, trace theorems have been extensively studied [9, 13, 14]. Feichtinger, Huang and Wang [9] considered the trace theorems on anisotropic modulation spaces $M_s^{p,q,r}$ with $0 < p, q, r < \infty, s \in \mathbb{R}$ and they obtained $\mathbb{T}M_s^{p,q,p \wedge q \wedge 1}(\mathbb{R}^n) = M_s^{p,q}(\mathbb{R}^{n-1})$. In [10, 15], we find that for $0 < p, q \leq \infty$, and $s - 1/p > (n-1)(1/p - 1)$, we have $\mathbb{T}B_s^{p,q}(\mathbb{R}^n) = B_{s-1/p}^{p,q}(\mathbb{R}^{n-1})$ and $\mathbb{T}F_s^{p,q}(\mathbb{R}^n) = F_{s-1/p}^{p,p}(\mathbb{R}^{n-1})$ (the case $F^{\infty,q}$ is omitted). The use of atoms as a framework in studying trace problems can be found in [15] and the references within.

Our main results are the following.

Theorem 1.1. *Let $n \geq 2, 0 < p, q < \infty, s \in \mathbb{R}$. Then*

$$\mathbb{T} : f(x) \rightarrow f(\bar{x}, 0), \quad \bar{x} = (x_1, x_2, \dots, x_{n-1})$$

is a trace-retraction from $W_s^{p,q,1 \wedge q}(\mathbb{R}^n)$ to $W_s^{p,q}(\mathbb{R}^{n-1})$.

In view of the embedding in Theorem 2.1 (II-ii), we immediately have the following corollary.

Corollary 1.1. *Let $n \geq 2, 0 < p, q < \infty, s \geq 0$. Then for any $\epsilon > 0$*

$$\mathbb{T} : W_{s+\frac{1}{1 \wedge q}-\frac{1}{q}+\epsilon}^{p,q}(\mathbb{R}^n) \rightarrow W_s^{p,q}(\mathbb{R}^{n-1}).$$

We remark that our result shows independence of p . This is due the pointwise estimates we were able to prove in Section 3. An interesting observation is that, the trace theorem of Triebel-Lizorkin spaces stated above, shows independence in q . This difference might be due to the decomposition operators used in the norm of each function spaces.

The paper is organised as follows: In Section 2, the embeddings between standard and anisotropic Wiener amalgam spaces are given. We also define notations, function spaces and some Lemmas to be used throughout this paper. In Section 3, we prove our main result, Theorem 1.1 and the sharpness of Corollary 1.1.

2. PRELIMINARIES

Notations. The Schwartz class of test functions on \mathbb{R}^n shall be denoted by $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ and its dual, the space of tempered distributions, by $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^n)$. The $L^p(\mathbb{R}^n)$ norm is given by $\|f\|_{L^p} = (\int_{\mathbb{R}^n} |f(x)|^p dx)^{1/p}$ whenever $1 \leq p < \infty$, and $\|f\|_{L^\infty} = \text{ess.sup}_{x \in \mathbb{R}^n} |f(x)|$. The Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is given by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i2\pi x \cdot \xi} f(x) dx$$

which is an isomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ onto itself that extends to the tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ by duality. The inverse Fourier transform is given by $\mathcal{F}^{-1}f(x) = \check{f}(x) = \int_{\mathbb{R}^n} e^{i2\pi \xi \cdot x} f(\xi) d\xi$. Given $1 \leq p \leq \infty$, we denote by p' the conjugate exponent of p (i.e. $1/p + 1/p' = 1$). We use the notation $u \lesssim v$ to denote $u \leq cv$

for a positive constant c independent of u and v . We write $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$. We now define the function spaces in this paper.

Let $\eta : \mathbb{R} \rightarrow [0, 1]$ be a smooth bump function satisfying

$$\eta(\xi) := \begin{cases} 1, & |\xi| \leq 1 \\ \text{smooth}, & 1 < |\xi| \leq 2 \\ 0, & |\xi| \geq 2. \end{cases}$$

We write for $k = (k_1, \dots, k_n)$ and $\xi = (\xi_1, \dots, \xi_n)$,

$$\phi_{k_i} = \eta(2(\xi_i - k_i)).$$

Put

$$(1) \quad \varphi_k(\xi) = \frac{\phi_{k_1}(\xi_1) \dots \phi_{k_n}(\xi_n)}{\sum_{k \in \mathbb{Z}^n} \phi_{k_1}(\xi_1) \dots \phi_{k_n}(\xi_n)}, \quad k \in \mathbb{Z}^n.$$

Definition 2.1 (*Wiener amalgam spaces*). For $0 < p, q \leq \infty$, and $s \in \mathbb{R}$, the Wiener amalgam space $W_s^{p,q}$ consists of all tempered distributions $f \in \mathcal{S}'$ for which the following is finite:

$$(2) \quad \|f\|_{W_s^{p,q}} = \| \{ \langle k \rangle^s \square_k f \} \|_{\ell^q} \|_{L^p},$$

with $\square_k f = \mathcal{F}^{-1}(\varphi_k \hat{f})$.

We note that (2) is a quasi-norm if $0 < p, q \leq \infty$, and norm if $1 \leq p, q \leq \infty$. Moreover, (2) is independent of the choice of $\varphi = \{\varphi_k\}_{k \in \mathbb{Z}^n}$. We refer the reader to [5, 6, 8] for equivalent definitions (continuous versions).

We write $\bar{x} = (x_1, x_2, \dots, x_{n-1})$ and define the anisotropic Wiener amalgam spaces $W^{p,q,r}$ by the following norm,

$$\|f\|_{W_s^{p,q,r}(\mathbb{R}^n)} = \| \left(\sum_{k_n \in \mathbb{Z}} \left(\sum_{\bar{k} \in \mathbb{Z}^{n-1}} \langle \bar{k} \rangle^{sq} |\square_k f|^q \right)^{r/q} \right)^{1/r} \|_{L^p(\mathbb{R}^n)}.$$

Similarly, for $\bar{\bar{x}} = (x_1, x_2, \dots, x_{n-2})$, we define

$$\|f\|_{W_s^{p,q,r,\bar{r}}(\mathbb{R}^n)} = \| \left(\sum_{(k_{n-1}, k_n) \in \mathbb{Z}^2} \left(\sum_{\bar{\bar{k}} \in \mathbb{Z}^{n-2}} \langle \bar{\bar{k}} \rangle^{sq} |\square_k f|^q \right)^{r/q} \right)^{1/r} \|_{L^p(\mathbb{R}^n)}.$$

Comparing amalgam spaces $W_s^{p,q}$ with anisotropic amalgam spaces $W_s^{p,q,r}$ we see that $W_s^{p,q}$ is, but $W_s^{p,q,r}$ is not rotational invariant. Using the almost orthogonality of φ we see that the $W_s^{p,q,r}$ is independent of φ . Moreover, recalling that $\|f\|_{W_s^{p,q,r}}$ is the function sequence $\{\square_k f\}_{k \in \mathbb{Z}^n}$ equipped with the $L^p \ell_{k_n}^r \ell_{\bar{k}}^q$ norm, it is easy to see that $W_s^{p,q,r}$ is a quasi-Banach space for any $s \in \mathbb{R}, p, q, r \in (0, \infty]$ and a Banach space for any $s \in \mathbb{R}, 1 \leq p, q, r \leq \infty$. Moreover, the Schwartz space is dense in $W_s^{p,q,r}$ if $p, q, r < \infty$. The proofs are similar to those of amalgam spaces in [5, 6, 8].

We collect properties of Wiener amalgam spaces in the following lemma.

Lemma 2.1. *Let $p, q, p_i, q_i \in [1, \infty]$ for $i = 1, 2$ and $s_j \in \mathbb{R}$ for $j = 1, 2$. Then*

- (1) $\mathcal{S}(\mathbb{R}^n) \hookrightarrow W^{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$;
- (2) \mathcal{S} is dense in $W^{p,q}$ if p and $q < \infty$;
- (3) If $q_1 \leq q_2$ and $p_1 \leq p_2$, then $W^{p_1, q_1} \hookrightarrow W^{p_2, q_2}$;

(4) If $s_1 \geq s_2$, then $W_{s_1}^{p,q} \hookrightarrow W_{s_2}^{p,q}$;

(5) (Complex interpolation) For $0 < \theta < 1$. Let $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$, $\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$ and $s = \theta s_1 + (1-\theta)s_2$. Then

$$[W_{s_1}^{p_1,q_1}, W_{s_2}^{p_2,q_2}]_{[\theta]} = W_s^{p,q}.$$

The proofs of these statements can be found in [5, 7, 8, 12].

Theorem 2.1 (Embedding: $W_s^{p,q} \hookrightarrow W_{s'}^{p,q,r}$). Let $p, q, r \in (0, \infty]$ and $s \geq 0$.

(I): The case $r = q$.

(I - i): The case $r = q = \infty$.

$$W_s^{p,\infty} \hookrightarrow W_s^{p,\infty,\infty}.$$

(I - ii): The case $r = q < \infty$.

$$W_s^{p,q} \hookrightarrow W_s^{p,q,q}.$$

(II): The case $r < q$.

(II - i): The case $q = \infty$. If $s > 1/r$, then

$$W_s^{p,\infty} \hookrightarrow W_{s'}^{p,\infty,r},$$

for any $s' \in (-\infty, s - 1/r)$.

(II - ii): The case $q < \infty$. If $s > (1/r - 1/q)$, then

$$W_s^{p,q} \hookrightarrow W_{s'}^{p,q,r},$$

for any $s' \in (-\infty, s - (1/r - 1/q))$.

(III): The case $q < r$.

(III - i): The case $r = \infty$.

$$W_s^{p,q} \hookrightarrow W_s^{p,q,\infty}.$$

(III - ii): The case $r < \infty$.

$$W_s^{p,q} \hookrightarrow W_s^{p,q,r}.$$

Proof. For part (I), it suffice to show the following estimates.

(I-i):

$$\sup_{k_n} \sup_{\bar{k}} \langle \bar{k} \rangle^s |\square_k f| \leq \sup_k \langle k \rangle^s |\square_k f|.$$

(I - ii):

$$\left(\sum_{k_n} \sum_{\bar{k}} \langle \bar{k} \rangle^{sq} |\square_k f|^q \right)^{1/q} \leq \left(\sum_k \langle k \rangle^{sq} |\square_k f|^q \right)^{1/q}.$$

(II - i): Let $s' := s - 1/r - \varepsilon$, ($\varepsilon > 0$). We may assume that $s' \geq 0$.

$$\left(\sum_{k_n} \sup_{\bar{k}} \langle \bar{k} \rangle^{s'r} |\square_k f|^r \right)^{1/r} \leq \left(\sup_k \langle k \rangle^s |\square_k f| \right) \times \left(\sum_{k_n} \sup_{\bar{k}} \langle k \rangle^{-sr} \langle \bar{k} \rangle^{s'r} \right)^{1/r}$$

The last term is equivalent to

$$\left(\sum_{m \in \mathbb{Z}} \left(\sup_{t \geq 1} \left(\frac{1}{t + |m|} \right)^s t^{s'} \right)^r \right)^{1/r} \leq \left(\sum_{m \in \mathbb{Z}} \left(\frac{1}{1 + |m|} \right)^{(s-s')r} \sup_{t \geq 1} t^{(s'-s')r} \right)^{1/r} < \infty,$$

here $(s - s')r = 1 + \varepsilon r > 1$ and $s' \geq 0$ have been used.

(II - ii): Let $s' := s - (1/r - 1/q) - \varepsilon$, ($\varepsilon > 0$). It suffice to show the embedding in the case $s' \geq 0$. Remark that $q/r \in (1, \infty)$ and $(q/r)' = 1/(r(1/r - 1/q))$. Let $\alpha := 1 - r/q + \varepsilon r$.

$$\begin{aligned} & \left[\sum_{k_n} \left\{ \sum_{\bar{k}} \langle \bar{k} \rangle^{s'q} |\square_k f|^q \right\}^{r/q} \right]^{1/r} \\ &= \left[\sum_{k_n} \left\{ \sum_{\bar{k}} \langle \bar{k} \rangle^{s'q} \langle k_n \rangle^{\alpha q/r} |\square_k f|^q \right\}^{r/q} \langle k_n \rangle^{-\alpha} \right]^{1/r} \\ &\leq \left[\left\{ \sum_{k_n} \sum_{\bar{k}} \langle \bar{k} \rangle^{s'q} \langle k_n \rangle^{\alpha q/r} |\square_k f|^q \right\}^{r/q} \times \left(\sum_{k_n} \langle k_n \rangle^{-\alpha(q/r)'} \right)^{1/(q/r)'} \right]^{1/r} \\ &\lesssim \left(\sum_k \langle \bar{k} \rangle^{s'q} \langle k_n \rangle^{\alpha q/r} |\square_k f|^q \right)^{1/q} \\ &= \left\{ \sum_k \langle k \rangle^{sq} |\square_k f|^q \left(\langle \bar{k} \rangle^{s'} \langle k_n \rangle^{\alpha/r} \langle k \rangle^{-s} \right)^q \right\}^{1/q} \\ &\leq \left[\sup_k \langle \bar{k} \rangle^{s'} \langle k_n \rangle^{\alpha/r} \langle k \rangle^{-s} \right] \times \left\{ \sum_k \langle k \rangle^{sq} |\square_k f|^q \right\}^{1/q}. \end{aligned}$$

Here, we have used $\alpha(q/r)' = 1 + \frac{\varepsilon}{1/r - 1/q} > 1$. Because $\alpha/r = 1/r - 1/q + \varepsilon = s - s'$, $s - s' \geq 0$ and $s' \geq 0$,

$$\langle \bar{k} \rangle^{s'} \langle k_n \rangle^{\alpha/r} \langle k \rangle^{-s} = \left(\frac{\langle k_n \rangle}{\langle k \rangle} \right)^{s-s'} \left(\frac{\langle \bar{k} \rangle}{\langle k \rangle} \right)^{s'} \lesssim 1.$$

(III - i):

$$\sup_{k_n} \left(\sum_{\bar{k}} \langle \bar{k} \rangle^{sq} |\square_k f|^q \right)^{1/q} \leq \left(\sum_k \langle k \rangle^{sq} |\square_k f|^q \right)^{1/q}.$$

Here, we have used $s \geq 0$.

(III - ii): Using the embedding $\ell^q \hookrightarrow \ell^r$,

$$\begin{aligned} \left\{ \sum_{k_n} \left(\sum_{\bar{k}} \langle \bar{k} \rangle^{sq} |\square_k f|^q \right)^{r/q} \right\}^{1/r} &\leq \left(\sum_k \langle \bar{k} \rangle^{sq} |\square_k f|^q \right)^{1/q} \\ &\leq \left(\sum_k \langle k \rangle^{sq} |\square_k f|^q \right)^{1/q}. \end{aligned}$$

In the last inequality, we need $s \geq 0$. □

Lemma 2.2 (Triebel, [14]). *Let $0 < p < \infty$ and $0 < q \leq \infty$. Let $\Omega = \{\Omega_k\}_{k \in \mathbb{Z}^n}$ be a sequence of compact subsets of \mathbb{R}^n . Let d_k be the diameter of Ω_k . If $0 < r < \min(p, q)$, then there exist a constant c such that*

$$\left\| \sup_{z \in \mathbb{R}^n} \frac{|f_k(\cdot - z)|}{1 + |d_k z|^{n/r}} \right\|_{L^p(\ell^q)} \leq c \|f_k\|_{L^p(\ell^q)}$$

holds for all $f \in L^p_\Omega(\ell^q)$, where $f = \{f_k\}$, $\|f_k\|_{L^p(\ell^q)} = \|\|f_k(\cdot)\|\|_{L^p}$ and

$$L^p_\Omega(\ell^q) = \{f \mid f = \{f_k\}_{k \in \mathbb{Z}^n} \subset \mathcal{S}', \text{supp } \mathcal{F}f_k \subset \Omega_k, \text{ and } \|f_k\|_{L^p(\ell^q)} < \infty\}.$$

Definition 2.2 (Maximal Functions). Let $b > 0$ and $f \in \mathcal{S}$. Then

$$(3) \quad \square_k^* f(x) := \sup_{y \in \mathbb{Z}^n} \frac{|\square_k f(x - y)|}{1 + |y|^b} \quad x \in \mathbb{R}^n, k \in \mathbb{Z}^n$$

Proposition 2.1. *Let $0 < p < \infty$ and $0 < q \leq \infty$, $b > \frac{n}{\min(p, q)}$. Then*

$$(4) \quad \left\| \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} |\square_k^* f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$

and

$$(5) \quad \left\| \left(\sum_{k_n \in \mathbb{Z}} \left(\sum_{\bar{k} \in \mathbb{Z}^{n-1}} \langle \bar{k} \rangle^{sq} |\square_k^* f|^q \right)^{r/q} \right)^{1/r} \right\|_{L^p(\mathbb{R}^n)}$$

are equivalent norms in $W_s^{p,q}(\mathbb{R}^n)$ and $W_s^{p,q,r}(\mathbb{R}^n)$, respectively.

The proof is a direct consequence of Lemma 2.2, taking $f_k = \square_k f$. See also [13, Proposition].

3. PROOF OF THE MAIN RESULTS

First, we narrate the idea of the proof. We give an equivalent formulation for $\square_{\bar{k}}(\mathbb{T}f)(\bar{x})$, a function in \mathbb{R}^{n-1} , via some $\square_{\bar{k},l} f(\bar{x}, 0)$ a function in \mathbb{R}^n . Then we compute for pointwise estimates between the corresponding ℓ^q norms and $\ell_{k_n}^r \ell_{\bar{k}}^q$ norms for cases $0 < q < 1$ and $1 \leq q < \infty$, separately. Finally, taking $L^p(\mathbb{R}^{n-1})$ norms and using our equivalent norms in Proposition (2.1), we arrive to our conclusion.

We denote $\mathcal{F}_{\bar{x}}(\mathcal{F}_{\bar{\xi}}^{-1})$ the partial (inverse) Fourier transform on \bar{x} ($\bar{\xi}$) $\in \mathbb{R}^{n-1}$. Write $\{\varphi_{\bar{k}}\}_{\bar{k} \in \mathbb{Z}^{n-1}}$ as versions of (1) in \mathbb{R}^{n-1} . By the support property of $\varphi_{\bar{k}}$, we observe

$$\begin{aligned}
 \square_{\bar{k}}(\mathbb{T}f)(\bar{x}) &= (\mathcal{F}_{\bar{\xi}}^{-1} \varphi_{\bar{k}} \mathcal{F}_{\bar{x}})(\mathbb{T}f)(\bar{x}) \\
 &= \sum_{l \in \mathbb{Z}^n} \{ \mathcal{F}_{\bar{\xi}}^{-1} \varphi_{\bar{k}} \mathcal{F}_{\bar{x}} [(\mathcal{F}^{-1} \varphi_l \mathcal{F} f)(\bar{y}, 0)] \}(\bar{x}) \\
 &= \sum_{l \in \mathbb{Z}^n} \chi_{(|\bar{k}-\bar{l}| \leq 1)} (\mathcal{F}^{-1} \psi_{\bar{k},l} \mathcal{F} f)(\bar{x}, 0) \\
 (6) \quad &= \sum_{l \in \mathbb{Z}^n} \chi_{(|\bar{k}-\bar{l}| \leq 1)} \square_{\bar{k},l} f(\bar{x}, 0),
 \end{aligned}$$

where $\psi_{\bar{k},l}(\xi) = \varphi_{\bar{k}}(\bar{\xi}) \varphi_l(\xi)$, $l = (\bar{l}, l_n)$, and $\square_{\bar{k},l} f := \mathcal{F}^{-1} \psi_{\bar{k},l} \mathcal{F} f$. Note that the left-hand side is a function in \mathbb{R}^{n-1} while the right-hand side is a function in \mathbb{R}^n .

Recall our maximal function (3) and take $y_1 = y_2 = \dots = y_{n-1} = 0, y_n = x_n$ we have for $|x_n| \leq 1$,

$$(7) \quad |\square_k f(\bar{x}, 0)| \lesssim \square_k^* f(x).$$

Proof of Theorem 1.1. We start by taking the ℓ^q -norm of (6). We write,

$$(8) \quad \left(\sum_{\bar{k} \in \mathbb{Z}^{n-1}} \langle \bar{k} \rangle^{sq} |\square_{\bar{k}}(\mathbb{T}f)(\bar{x})|^q \right)^{1/q} = \left(\sum_{\bar{k} \in \mathbb{Z}^{n-1}} \langle \bar{k} \rangle^{sq} \left(\sum_{l \in \mathbb{Z}^n} \chi_{(|\bar{k}-\bar{l}| \leq 1)} \square_{\bar{k},l} f(\bar{x}, 0) \right)^q \right)^{1/q}.$$

For $0 < q < 1$, we estimate (8) by

$$\begin{aligned}
 \left(\sum_{\bar{k} \in \mathbb{Z}^{n-1}} \langle \bar{k} \rangle^{sq} |\square_{\bar{k}}(\mathbb{T}f)(\bar{x})|^q \right)^{1/q} &\lesssim \left(\sum_{l \in \mathbb{Z}^n} \sum_{\bar{k} \in \mathbb{Z}^{n-1}} \langle \bar{l} \rangle^{sq} \chi_{(|\bar{k}-\bar{l}| \leq 1)} |\square_{\bar{k},l} f(\bar{x}, 0)|^q \right)^{1/q} \\
 (9) \quad &= \left(\sum_{l_n \in \mathbb{Z}} \sum_{\bar{l} \in \mathbb{Z}^{n-1}} \langle \bar{l} \rangle^{sq} \sum_{\bar{k} \in \mathbb{Z}^{n-1}} \chi_{(|\bar{k}-\bar{l}| \leq 1)} |\square_{\bar{k},l} f(\bar{x}, 0)|^q \right)^{1/q}
 \end{aligned}$$

Note that $\sum_{\bar{k} \in \mathbb{Z}^{n-1}} \chi_{(|\bar{k}-\bar{l}| \leq 1)} |\square_{\bar{k},l} f(\bar{x}, 0)|^q = \sum_{j=1}^{n-1} |\square_{\bar{l} \pm e_j, l} f(\bar{x}, 0)|^q$, where e_j is the j^{th} column of the identity matrix. In the sequel, it suffice to consider only the case $j = 1$. Moreover, we write $\tilde{\square}_l f := \square_{\bar{l} \pm e_1, l} f$ for some ψ_l satisfying (1). Using (7) we have,

$$(10) \quad \left(\sum_{l_n \in \mathbb{Z}} \sum_{\bar{l} \in \mathbb{Z}^{n-1}} \langle \bar{l} \rangle^{sq} |\square_{\bar{l} \pm e_1, l} f(\bar{x}, 0)|^q \right)^{1/q} \lesssim \left(\sum_{l_n \in \mathbb{Z}} \sum_{\bar{l} \in \mathbb{Z}^{n-1}} \langle \bar{l} \rangle^{sq} |\tilde{\square}_l^* f(\bar{x}, x_n)|^q \right)^{1/q}.$$

Combining (9) and (10), then taking the $L^p(\mathbb{R}^{n-1})$ -norm and raising to p -th power gives,

$$\|f(\bar{x}, 0)\|_{W_s^{p,q}(\mathbb{R}^{n-1})}^p \lesssim \left\| \left(\sum_{l \in \mathbb{Z}^n} \langle \bar{l} \rangle^{sq} \tilde{\square}_l^{*q} f(\bar{x}, x_n) \right)^{1/q} \right\|_{L^p(\mathbb{R}^{n-1})}^p$$

Integrating over $x_n \in [0, 1]$,

$$\begin{aligned} \|f(\bar{x}, 0)\|_{W_s^{p,q}(\mathbb{R}^{n-1})} &\lesssim \left\| \left(\sum_{l \in \mathbb{Z}^n} \langle \bar{l} \rangle^{sq} \tilde{\square}_l^{*q} f(\bar{x}, x_n) \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \|f\|_{W_s^{p,q,q}(\mathbb{R}^n)}. \end{aligned}$$

Note that the last inequality follows from Proposition 2.1.

For $1 \leq q \leq \infty$, we use Minkowski's inequality to give an upper bound of (8) as follows,

$$\begin{aligned} \left(\sum_{\bar{k} \in \mathbb{Z}^{n-1}} \langle \bar{k} \rangle^{sq} |\square_{\bar{k}}(\mathbb{T}f)(\bar{x})|^q \right)^{1/q} &\lesssim \left(\sum_{\bar{k}, \bar{l} \in \mathbb{Z}^{n-1}} \langle \bar{k} \rangle^{sq} \left(\sum_{l_n \in \mathbb{Z}} \chi_{(|\bar{k}-\bar{l}| \leq 1)} \square_{\bar{k}, l} f(\bar{x}, 0) \right)^q \right)^{1/q} \\ &\lesssim \sum_{l_n \in \mathbb{Z}} \left(\sum_{\bar{k}, \bar{l} \in \mathbb{Z}^{n-1}} \langle \bar{k} \rangle^{sq} \chi_{(|\bar{k}-\bar{l}| \leq 1)} |\square_{\bar{k}, l} f(\bar{x}, 0)|^q \right)^{1/q} \\ (11) \quad &\lesssim \sum_{l_n \in \mathbb{Z}} \left(\sum_{\bar{l} \in \mathbb{Z}^{n-1}} \langle \bar{l} \rangle^{sq} |\tilde{\square}_l^{*q} f(\bar{x}, x_n)|^q \right)^{1/q}. \end{aligned}$$

Repeating the arguments above on (11) gives us the estimate,

$$\|f(\bar{x}, 0)\|_{W_s^{p,q}(\mathbb{R}^{n-1})} \lesssim \|f\|_{W_s^{p,q,1}(\mathbb{R}^n)}.$$

Hence, we arrive to our desired estimates.

Let $\eta' \in \mathcal{S}(\mathbb{R})$ be a function with $\text{supp } \eta' \subset (-1/4, 1/4)$ and $(\mathcal{F}_{\xi_n}^{-1})\eta'(0) = 1$. For any $f \in W_s^{p,q}(\mathbb{R}^{n-1})$, we define $g(x) = (\mathbb{T}^{-1}f)(x) := [(\mathcal{F}_{\xi_n}^{-1})\eta'(x_n)] f(\bar{x})$. We easily see that $g(\bar{x}, 0) = f(\bar{x})$ and $\square_k g = 0$ when $|k_n| \geq 3$. Moreover, we can decompose $\square_k g = \square_{\bar{k}} f \cdot \square_{k_n} (\mathcal{F}_{\xi_n}^{-1}\eta')$ due to the way φ_k is defined in (1). Now we do an estimate,

$$\begin{aligned} \|g\|_{W_s^{p,q,q \wedge 1}(\mathbb{R}^n)} &= \left\| \left(\sum_{k_n \in \mathbb{Z}} \left(\sum_{\bar{k} \in \mathbb{Z}^{n-1}} \langle \bar{k} \rangle^{sq} |\square_{\bar{k}} g|^q \right)^{q \wedge 1/q} \right)^{1/q \wedge 1} \right\|_{L^p(\mathbb{R}^n)} \\ &= \left\| \left(\sum_{\bar{k} \in \mathbb{Z}^{n-1}} \langle \bar{k} \rangle^{sq} |\square_{\bar{k}} f|^q \right)^{1/q} \left(\sum_{|k_n| \leq 2} |\square_{k_n} (\mathcal{F}_{\xi_n}^{-1}\eta')|^{1 \wedge q} \right)^{1/q \wedge 1} \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \|f\|_{W_s^{p,q}(\mathbb{R}^{n-1})}. \end{aligned}$$

Thus, $\mathbb{T}^{-1} : W_s^{p,q}(\mathbb{R}^{n-1}) \rightarrow W_s^{p,q,q \wedge 1}(\mathbb{R}^n)$.

□

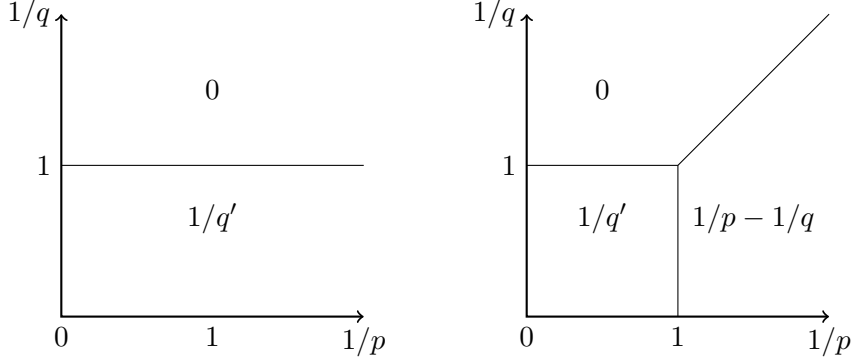


FIGURE 1. Comparison between the critical regularity index s for $\mathbb{T}W_{s+\epsilon}^{p,q}(\mathbb{R}^n) = W^{p,q}(\mathbb{R}^{n-1})$ (left) and $\mathbb{T}M_{s+\epsilon}^{p,q}(\mathbb{R}^n) = M^{p,q}(\mathbb{R}^{n-1})$ (right).

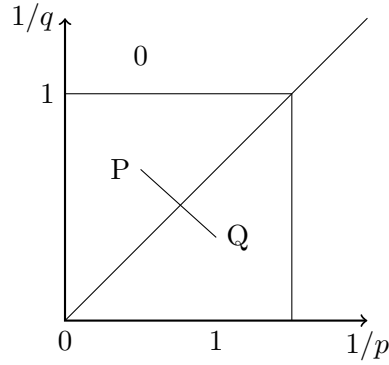


FIGURE 2. Contradiction argument using interpolation.

As the end of this paper, we discuss the optimality of Corollary 1.1. We recall the counterexample given in [9]. For $1 < p, q < \infty$, there exist a function which shows

$$\mathbb{T} : M_{1/q'}^{p,q}(\mathbb{R}^n) \not\rightarrow M_0^{p,q}(\mathbb{R}^{n-1}).$$

Since $M^{q,q} = W^{q,q}$, we also have $\mathbb{T} : W_{1/q'}^{q,q}(\mathbb{R}^n) \not\rightarrow W_0^{q,q}(\mathbb{R}^{n-1})$. Hence, Corollary 1.1 is sharp for $p = q, 1 < p, q < \infty$ (refer to FIGURE 1). We now claim that it is also sharp for all $1 < p, q < \infty$. Contrary to our claim, suppose $s = 1/q'$ implies $\mathbb{T}W_s^{p,q}(\mathbb{R}^n) = W^{p,q}(\mathbb{R}^{n-1})$. Then, by interpolation with the estimate for a point $Q(p_1, q_1)$ with $s = 1/q'_1$, one would obtain an improvement for the segment connecting $P(p, q)$ and $Q(p_1, q_1)$ (refer to FIGURE 2), which is not possible.

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REFERENCES

- [1] A. Bényi, K. Gröchenig, K. Okoudjou, and L. Rogers. Unimodular Fourier multipliers for modulation spaces. *J. Funct. Anal.*, 246(2):366–384, 2007.
- [2] E. Cordero and F. Nicola. Remarks on Fourier multipliers and applications to wave equation. *J. Math. Anal. Appl.*, 353:583–591, 2009.
- [3] J. Cunanan, M. Sugimoto, and M. Kobayashi. Inclusion relations between L^p -Sobolev and Wiener amalgam spaces. *J. Funct. Anal.*, 419(2):738–747, 2014.
- [4] L. Evans. *Partial differential equations*. Providence, R.I.: American Mathematical Society, 1998.
- [5] H. G. Feichtinger. Banach convolution algebras of wiener’s type. In *Proc. Conf. "Function, Series, Operators"*, Budapest, *Colloq. Math. Soc. János Bolyai*, pages 509–524. North-Holland, Amsterdam, 1980.
- [6] H. G. Feichtinger. Banach spaces of distributions of wiener type and interpolation. In *Functional Analysis and Approximation*, volume 60, pages 153–165. Birkhäuser Basel, 1981.
- [7] H. G. Feichtinger. Modulation spaces on locally compact abelian groups. Technical report, University of Vienna, 1983.
- [8] H. G. Feichtinger. Generalized amalgams, with applications to Fourier transform. *Canad. J. Math.*, 42(3):395–409, 1990.
- [9] H. G. Feichtinger, C. Huang, and B. Wang. Trace operators for modulation, α -modulation and besov spaces. *Applied and Computational Harmonic Analysis*, 30(1):110 – 127, 2011.
- [10] M. Frazier and B. Jawerth. Decomposition of besov spaces. *Indiana Univ. Math. J.*, 34:777–799, 1985.
- [11] K. Gröchenig and C. Heil. Modulation spaces and pseudodifferential operators. *Integral Equations Operator Theory*, 34(4):439–457, 1999.
- [12] J. Toft. Continuity properties for modulation spaces, with applications to pseudo-differential calculus-II. *Ann. Global Anal. Geom.*, 26:73–106, 2004.
- [13] H. Triebel. Modulation spaces on the euclidean n-space. *Z. Anal. Anwendungen*, 2:443–457, 1983.
- [14] H. Triebel. *Theory of function spaces*. Birkhäuser-Verlag, 1983.
- [15] H. Triebel. *Theory of function spaces II*. Birkhäuser, 1992.
- [16] B. Wang and C. Huang. Frequency-uniform decomposition method for generalized BO, KdV and NLS equations. *J. Differential Equations*, 239:213–250, 2007.

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